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AUTOMATIC CONTINUITY IN HOMEOMORPHISM GROUPS OF COMPACT 2-MANIFOLDS

BY

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ABSTRACT

We show that any homomorphism from the homeomorphism group of a compact 2-manifold, with the compact-open topology, or equivalently, with the topology of uniform convergence, into a separable topological group is automatically continuous.

1. Introduction

A number of results have surfaced in recent years that intimately connect topologies on transformation groups with the underlying group structure. Of course, many classical mathematical results, variously formulated as **rigidity** or **reconstruction** results, can be viewed in this way. Namely, as saying that if G is the group of transformations of some mathematical object K, then K can be completely recovered within its category from G as an abstract group, and hence any natural transformation group topology on G is also given by the abstract group G. Related to this are results saying that any automorphism of G is inner and hence given by a transformation of K.

However, recently there have been indications that certain topological groups might not only be determined by the underlying abstract group, but, in fact, that the topology is also preserved under homomorphisms. Some indications of this come from the so-called **small index property** for separable, complete

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metric groups saying that a subgroup of index $< 2^{\aleph_0}$ is open. This implies that any homomorphism into the group $\mathcal{S}_\infty$ of all permutations of $\mathbb{N}$ is continuous, when the latter has been equipped with the topology of pointwise convergence on the discrete set $\mathbb{N}$. This follows from the fact that the topology of $\mathcal{S}_\infty$ is generated by its open subgroups. The small index property has now been proved for a great number of closed subgroups of $\mathcal{S}_\infty$ itself (perhaps the most general result is due to Hodges, Hodkinson, Lascar, and Shelah [HHLS93]), but it also holds for certain groups that are not topologically isomorphic to subgroups of $\mathcal{S}_\infty$, e.g., Homeo(S^1) [RoSo07].

Nevertheless, these results put rather heavy restrictions on the target groups, namely, their topology has to be given by the open subgroups. This condition was removed by Kechris and the author in [KeRo07], in which it was shown that for many closed subgroups of $\mathcal{S}_\infty$ one has a completely general result of **automatic continuity**, namely, that any homomorphism from one of these groups into a separable topological group is continuous. This line of research was continued by Solecki and the author in [RoSo07] in which this property was verified for many other groups including Homeo(S^1). Thus, one could hope for this to be true for a general class of homeomorphism groups of manifolds, and we shall provide the first step here by considering manifolds of dimension 2.

Automatic continuity turns out to have connections with other dynamical properties of groups and, for example, has provided the only known examples of discrete groups with the so-called fixed point on metric compacta property, i.e., discrete groups all of whose actions on compact metric spaces have a fixed point. We shall not develop any of these relations here, but only refer the reader to [RoSo07] for more on this.

It is well-known and easy to see that for any compact metric space (X, d), its group of homeomorphisms is a separable complete metric group when equipped with the topology of uniform convergence, or equivalently, with the compact open topology. In fact, a compatible right-invariant metric on Homeo(X, d) is given by $d_\infty(g, f) = \sup_{x\in X} d(g(x), f(x))$, and a complete metric by $d'_\infty(g, f) = d_\infty(g, f) + d_\infty(g^{-1}, f^{-1})$. We denote by $B(x, \epsilon)$ the open ball of radius ϵ around x and by $\overline{B}(x, \epsilon)$ the corresponding closed ball.

If $g \in$ Homeo(X, d), $\mathrm{supp}^\circ(g) = \{x \in X : g(x) \neq x\}$ and $\mathrm{supp}(g)$ is its closure, which is called the **support** of g.

We intend to show here that in the case of compact 2-manifolds, this group topology is intrinsically given by the underlying discrete or abstract group, in the sense that any homomorphism π from this group into a separable group is continuous.

Theorem 1.1: *Let M be a compact 2-manifold and $\pi : \mathrm{Homeo}(M) \to H$ a homomorphism into a separable group. Then π is automatically continuous when* Homeo(M) *is equipped with the compact-open topology.*

Let us first note the following fact, which follows easily from known results and helps to clearify the situation.

Proposition 1.2: *Suppose G is a topological group. Then the following conditions are equivalent.*

(1) *Any homomorphism $\pi : G \to \mathrm{Homeo}([0,1]^{\mathbb{N}})$ is continuous;*
(2) *any homomorphism $\pi : G \to H$ into a separable group is continuous.*

Proof. As $[0,1]^{\mathbb{N}}$ is a compact metric space, its homeomorphism group is a (completely metrisable) separable group in the compact-open topology, so (1) is a special case of (2).

For the other implication, suppose that (1) holds and let H be separable. Let N be the closed normal subgroup of H consisting of all elements that cannot be separated from the identity by an open set. Let H/N be the quotient topological group, which is Hausdorff and separable, and, in particular, any non-empty open set covers the group by countably many translates. However, it is an old result (see I.I. Guran [Gu81]) that for Hausdorff groups this condition is equivalent to being topologically isomorphic to a subgroup of a direct product of separable metric groups, or equivalently, second countable Hausdorff groups (by the Birkhoff–Kakutani metrisation Theorem). Also, a result of Uspenskiĭ [Us86] states that any separable metric group is topologically isomorphic to a subgroup of $\mathrm{Homeo}([0,1]^{\mathbb{N}})$, and we can therefore see H/N as a subgroup of some power of $\mathrm{Homeo}([0,1]^{\mathbb{N}})$. Thus, as a mapping into the Tikhonov product is continuous if and only if the composition with each coordinate projection is continuous, π composed with the quotient mapping is continuous, and hence by the choice of N, π is also continuous. ∎

However, we shall not make use of this result, but rather simplify matters by using arbitrary subsets of the group satisfying a certain algebraic largeness condition (instead of working with arbitrary homomorphisms). Let G be a group and $W \subseteq G$ be a symmetric set. We say that W is **countably syndetic** if there are countably many left-translates of W whose union cover G. Moreover, if G is a topological group, we say that G is **Steinhaus** if for some $k \geq 1$ and all symmetric, countably syndetic $W \subseteq G$, $\mathrm{Int}(\mathrm{W}^{\mathrm{k}}) \neq \emptyset$. It is not hard to prove (see, e.g., [RoSo07]) that Steinhaus groups satisfy the equivalent conditions of Proposition 1.2, and this is the condition that we will verify. Note, however, the order of quantification; the k is universal for all symmetric, countably syndetic W. Indeed, the group $\mathrm{Homeo}_+(S^1)$ equipped with the trivial topology $\tau = \{\emptyset, \mathrm{Homeo}_+(S^1)\}$ satisfies the condition when we have reversed the quantifiers, but the identity homomorphism into itself equipped with the compact-open topology is obviously discontinuous.

It is instructive to note from which groups one can construct discontinuous homeomorphisms. Of course, the first case that comes to mind is $(\mathbb{R}, +)$, on which one can with the help of a Hamel basis, i.e., a basis for $\mathbb{R}$ as a $\mathbb{Q}$-vector space, construct discontinuous automorphisms, and, in fact, construct group isomorphisms between $\mathbb{R}$ and $\mathbb{R}^2$. Also if $G = \prod_n F_n$, where the F_n are finite non-trivial groups, satisfies automatic continuity, then $|F_n| \to \infty$. For otherwise, there is some infinite set $A \subseteq \mathbb{N}$ such that $F_n = F_m$ for all $n, m \in A$. Let $\mathcal{U}$ be a non-principal ultrafilter on A and set $H = \{g \in G \bigm| \{n \in A \bigm| g_n = 1\} \in \mathcal{U}\}$. Then H is a non-open subgroup of G of finite index and hence G has a discontinuous homomorphism into a finite group.

We finish this introduction by mentioning a few of the most interesting questions concerning automatic continuity.

Question 1.3: (1) Is there a compact metrisable group satisfying automatic continuity, i.e., satisfying the equivalent conditions of Proposition 1.2?
(2) What about a locally compact second countable group?
(3) Does the unitary group of separable infinite-dimensional Hilbert space $U(\ell_2)$ satisfy automatic continuity?
(4) Is Theorem 1.1 true for an arbitrary compact manifold M?
(5) What about compact triangulable manifolds?

Questions (1) and (2) would be a way of producing discrete groups acting faithfully on separable metric spaces, but such that all of their actions have compact, respectively, σ-compact orbits. This would be a strenthening in the separable case of the so called **Bergman** or **strong boundedness** property of a group, saying that any isometric action on a (not necessarily separable) metric space has bounded orbits. This property is known to hold for a large class of groups, e.g., $\mathcal{S}_\infty$ [Be06], Homeo(S^n) [CaFrCo06], and $U(\ell_2)$ [RiRo07]. I conjecture that the profinite group $\prod_n \mathrm{Alt}(2^n)$ should satisfy automatic continuity. The proofs given by Saxl, Shelah, and Thomas in [SaShTh96, Th99] go a far way in order to establish this and with a little extra work, one can make their proofs show also the Bergman property for $\prod_n \mathrm{Alt}(2^n)$. However, so far I have not been able to make it show that $\prod_n \mathrm{Alt}(2^n)$ is Steinhaus and thus that it satisfies automatic continuity.

Case (3) would, in conjunction with a result of Gromov and Milman [GrMi83], imply that $U(\ell_2)$ has the fixed point on metric compacta property as a discrete group.

As can be seen from the proof of Theorem 1.1 below, certain parts of the proof transfer directly to higher dimensional triangulable manifolds. Unfortunately, this is not the case throughout and one naturally wonders what happens for homeomorphism groups of higher dimensional manifolds. Geometric topology in higher dimensions is well developed and some of the work done around the annulus conjecture is certainly relevant here. However, the annulus conjecture by itself is not enough and it is for this reason that we have been forced to use ad hoc constructions based on Schönflies' Theorem to get the exact lemmas we need.

2. The proof

2.1. Commutators. We shall first prove a general lemma about homeomorphisms of $\mathbb{R}^n$.

Lemma 2.1: *Suppose that* $g \in \mathrm{Homeo}(\mathbb{R}^n)$ *has compact support. Then there are* $f, h \in \mathrm{Homeo}(\mathbb{R}^n)$ *with compact support such that* $g = [f,h] = fhf^{-1}h^{-1}$.

Proof. Fix some open ball $U_0 \subseteq \mathbb{R}^n$ containing the support of g and let (U_m) be a sequence of disjoint open balls such that for some distinct x_0 and x_1 in $\mathbb{R}^n$, the sequences $(\overline{U}_m)_{m\geq 0}$ and $(\overline{U}_{-m})_{m\geq 0}$ converge in the Vietoris topology to x_0 and x_1 respectively. We can now find a shift $h \in \mathrm{Homeo}(\mathbb{R}^n)$ with compact support, i.e., such that $h[U_m] = U_{m+1}$ and define f by letting $f|U_m = h^m g h^{-m}|U_m$ for $m \geq 0$ and setting $f = \mathrm{id}$ everywhere else. We now see that

$$hf^{-1}h^{-1}|U_m = h(h^{m-1}g^{-1}h^{-m+1})h^{-1}|U_m = h^m g^{-1} h^{-m}|U_m, \quad \text{for } m > 0$$

and

$$hf^{-1}h^{-1}|U_m = h\,\mathrm{id}\,h^{-1}|U_m = \mathrm{id}|U_m, \quad \text{for } m \leq 0,$$

while $hf^{-1}h^{-1} = \mathrm{id}$ everywhere else. Therefore, $f \cdot hf^{-1}h^{-1}|U_m = \mathrm{id}|U_m$ for $m > 0$, $f \cdot hf^{-1}h^{-1}|U_0 = f|U_0 = g|U_0$, $f \cdot hf^{-1}h^{-1}|U_m = \mathrm{id}|U_m$ for $m < 0$, and $fhf^{-1}h^{-1} = \mathrm{id}$ everyhere else. This shows that $g = [f,h] = fhf^{-1}h^{-1}$. ∎

Notice that in the proof above we use f and h with slightly bigger support than g. This leads to the question of whether every homeomorphism that fixes the boundary pointwise can be written as a commutator of two homeomorphisms that also fix the boundary pointwise. Also, what happens if we replace pointwise by setwise? Let us mention that the first question has a positive answer in dimension 1 as, for example, the group of orientation preserving homeomorphisms of $[0,1]$ has a comeagre conjugacy class [KuTr00]. The above result slightly strengthens a result of Mather [Ma71] saying that the homology groups of the group of homeomorphisms $\mathbb{R}^n$ with compact support vanish. One can of course also extend the lemma to $[0,\infty[\times\mathbb{R}^{n-1}$ and thus also improve the result of Rybicki [Ry96].

2.2. Countably syndetic sets. We will now prove some properties of countably syndetic sets in the homeomorphism groups of arbitrary manifolds. These results will allow us to solve our problem for compact two-dimensional manifolds and provide techniques for higher dimensions. So let M be a manifold of dimension n and fix a compatible complete metric d on M.

In the following we fix a countably syndetic symmetric subset $W \subseteq \mathrm{Homeo}(M)$ and a sequence $k_m \in \mathrm{Homeo}(M)$ such that $\bigcup_m k_m W = \mathrm{Homeo}(M)$.

Lemma 2.2: *For all distinct $y_1, \ldots, y_p \in M$ and $\epsilon > 0$, there are $\epsilon > \delta > 0$ and $z_i \in B(y_i, \epsilon)$ such that if $g \in \mathrm{Homeo}(M)$ has support contained in $D = \bigcup_{i=1}^{p} \overline{B}(z_i, \delta)$, then $g \in W^{16}$.*

Proof. Notice that it is enough to find $z_i \in B(y_i, \epsilon)$ and open neighbourhoods U_i of z_i such that if $g \in \mathrm{Homeo}(M)$ has support contained in $\bigcup_i U_i$, then $g \in W^{16}$. We choose some open neighbourhood of y_i, $E_i \subseteq B(y_i, \epsilon)$, that is homeomorphic to $]0,\epsilon[^n$. We also suppose that the sets E_i are 4ϵ-separated. We will also temporarily transport the standard euclidian metric from $]0,\epsilon[^n$ to each of the sets E_i. As we will be working separately on each of E_i, this will not cause a problem. Thus in the following, the notation $B(x, \beta)$ will refer to the balls in the transported euclidian metric, which we denote by d.

Sublemma 2.3: *For all $u_i \in E_i$ and $\gamma > 0$ such that $d(u_i, \partial E_i) > 2\gamma$, there are $\gamma > \alpha > 0$ and $x_i \in \partial B(u_i, \gamma)$ such that if $g \in \mathrm{Homeo}(M)$ has support contained in $A = \bigcup_{i=1}^{p} \overline{B}(x_i, \alpha) \cap \overline{B}(u_i, \gamma)$, then there is an $h \in W^2$ with support contained in $\bigcup_{i=1}^{p} \overline{B}(u_i, \gamma)$ such that $g|A = h|A$.*

Proof. Suppose that $u_1, \ldots, u_p$ is given. We fix for each $i \leq p$ a sequence of distinct points $x_m^i \in \partial B(u_i, \gamma)$ converging to some point $x_\infty^i \in \partial B(u_i, \gamma)$ and choose a sequence $\gamma/2 > \alpha_m > 0$ such that $B(x_m^i, \alpha_m) \cap B(x_l^i, \alpha_l) = \emptyset$ for any $m \neq l$ and all $i \leq p$. Thus, as $\alpha_m \to 0$, we have that if $g_m \in \mathrm{Homeo}(M)$ has support only in

$$A_m = \big(\overline{B}(x_m^1, \alpha_m) \cap \overline{B}(u_1, \gamma)\big) \cup \cdots \cup \big(\overline{B}(x_m^p, \alpha_m) \cap \overline{B}(u_p, \gamma)\big)$$

for each $m \geq 0$, then there is a homeomorphism $g \in \mathrm{Homeo}(M)$, whose support is contained in $C = \overline{B}(u_1, \gamma) \cup \cdots \cup \overline{B}(u_p, \gamma)$, such that $g|A_m = g_m|A_m$. We claim that for some $m_0 \geq 0$, if $g \in \mathrm{Homeo}(M)$ has support contained in A_{m_0}, then there is an element $h \in k_{m_0}W$, with support contained in C, such that $g|A_{m_0} = h|A_{m_0}$. Assume toward a contradiction that this is not the case. Then for every m we can find some $g_m \in \mathrm{Homeo}(M)$ with support contained in A_m such that for all $h \in k_m W$, if $\mathrm{supp}(h) \subseteq C$, then $g_m|A_m \neq h|A_m$. But then letting $g \in \mathrm{Homeo}(M)$ have support in C and agree with each g_m on A_m for each m, we see that if $h \in k_m W$ has support in C, then g disagrees with h on A_m. Therefore, g cannot belong to any $k_m W$, contradicting that these cover $\mathrm{Homeo}(M)$.

Suppose that m_0 has been chosen as above and denote $x_{m_0}^i$ by x_i, A_{m_0} by A, and α_{m_0} by α. Then for any $g \in \mathrm{Homeo}(M)$ with support contained in A, there is an element $h \in W^2$ with support contained in C such that $g|A = h|A$ for all $i \leq p$. To see this, it is enough to notice that we can find $h_0, h_1 \in k_{m_0}W$, with $\mathrm{supp}(h_0), \mathrm{supp}(h_1) \subseteq C$, such that $g|A = h_1|A$ and $\mathrm{id}|A = h_0|A$. But then $h_0^{-1}h_1 \in (k_{m_0}W)^{-1}k_{m_0}W = W^{-1}W = W^2$ and $g|A = \mathrm{id}\, g|A = h_0^{-1}h_1|A$. ∎

We will first apply Sublemma 2.3 to the situation where $u_i = y_i$ and $\gamma > 0$ is sufficiently small. We thus obtain $\gamma > \alpha > 0$ and $x_i \in \partial B(y_i, \gamma)$ such that if $g \in \mathrm{Homeo}(M)$ has support contained in $A = \bigcup_{i=1}^{p} \overline{B}(x_i, \alpha) \cap \overline{B}(y_i, \gamma)$, then there is an $h \in W^2$ with support contained in $\bigcup_{i=1}^{p} \overline{B}(y_i, \gamma)$ such that $g|A = h|A$.

Now, pick $y_i' \in B(x_i, \alpha) \cap B(y_i, \gamma)$ and $\gamma' > 0$ such that

$$B(y_i', 2\gamma') \subseteq B(x_i, \alpha) \cap B(y_i, \gamma).$$

We now apply Lemma 2.3 once again to this new situation, in order to obtain $\gamma' > \alpha' > 0$ and $x_i' \in \partial B(y_i', \gamma')$ such that if $g \in \mathrm{Homeo}(M)$ has support contained in $A' = \bigcup_{i=1}^p \overline{B}(x_i', \alpha') \cap \overline{B}(y_i', \gamma')$, then there is an $h \in W^2$ with support contained in $\bigcup_{i=1}^p \overline{B}(y_i', \gamma')$ such that $g|A' = h|A'$.

Now clearly there is a homeomorphism $a \in \mathrm{Homeo}(M)$ whose support is contained in $A = \bigcup_{i=1}^p \overline{B}(x_i, \alpha) \cap \overline{B}(y_i, \gamma)$ such that $a[B(y_i', \gamma')] = B(x_i', \alpha')$ and

$$a[\overline{B}(y_i', \gamma') \cap \overline{B}(x_i', \alpha')] = \overline{B}(y_i', \gamma') \cap \overline{B}(x_i', \alpha'),$$

and hence we can also find such an a in W^2, except that its support may now be all of $\bigcup_{i=1}^p \overline{B}(y_i, \gamma)$.

We therefore have that if $g \in \mathrm{Homeo}(M)$ has support contained in A', then $a^{-1}ga$ also has support contained in A', and so there is an $h \in W^2$ with support contained in $\bigcup_{i=1}^p \overline{B}(y_i', \gamma')$ such that $a^{-1}ga|A' = h|A'$. But then $g|A' = aha^{-1}|A'$, while

$$\mathrm{supp}(aha^{-1}) = a[\mathrm{supp}(h)] \subseteq a\Big[\bigcup_{i=1}^p \overline{B}(y_i', \gamma')\Big] = \bigcup_{i=1}^p \overline{B}(x_i', \alpha').$$

We now notice that $aha^{-1} \in W^6$, and thus that if $g \in \mathrm{Homeo}(M)$ has support contained in $A' = \bigcup_{i=1}^p \overline{B}(x_i', \alpha') \cap \overline{B}(y_i', \gamma')$, then there is some $f \in W^6$ with support contained in $\bigcup_{i=1}^p \overline{B}(x_i', \alpha')$ such that $g|A' = f|A'$.

Finally, suppose that $g \in \mathrm{Homeo}(M)$ is any homeomorphism having support contained in $\bigcup_{i=1}^p B(x_i', \alpha') \cap B(y_i', \gamma')$. Since the sets $B(x_i', \alpha') \cap B(y_i', \gamma')$ are homeomorphic to $\mathbb{R}^n$, working separately on each of these sets and noticing that g has compact support, we can invoke Lemma 2.1 to write g as a commutator $[b, c]$ for some $b, c \in \mathrm{Homeo}(M)$ whose supports are contained in $\bigcup_{i=1}^p B(x_i', \alpha') \cap B(y_i', \gamma') \subseteq A'$. Find now $h \in W^2$ agreeing with b on A' and with support contained in $\bigcup_{i=1}^p \overline{B}(y_i', \gamma')$, and, similarly, find $f \in W^6$ agreeing with c on A' and with support contained in $\bigcup_{i=1}^p \overline{B}(x_i', \alpha')$. Then the set of common support of h and f is included in A' on which they agree with b and c respectively, and we have therefore that $[h, f] = hfh^{-1}f^{-1} = bcb^{-1}c^{-1} = g$. In other words, $g \in W^{16}$. We can therefore finish the proof by choosing some $z_i \in B(x_i', \alpha') \cap B(y_i', \gamma')$ and letting $U_i = B(x_i', \alpha') \cap B(y_i', \gamma')$. ∎

2.3. Circular orders. In order to simplify notation, we will consider **circular orders** on finite sets. For x, y, z distinct points on S^1, y is said to be between x and z, in symbols $B(x, y, z)$, if going counterclockwise around S^1 from x to y one does not pass through z. A **finite circular order** is just a ternary relation R on a finite set that is isomorphic to B restricted to a finite subset of S^1. When R is a circular order on a finite set $\mathbb{F}$, we denote for each $x \in \mathbb{F}$ its immediate successor and immediate predecessor, i.e., the first elements encountered by going respectively counterclockwise and clockwise around $\mathbb{F}$, by x^+ and x^-. So, e.g., $(x^+)^- = x$.

2.4. A quantitative annulus theorem. The proof of our result is tightly connected with the methods of geometric topology related to the annulus theorem. However, the annulus theorem in itself will not suffice in our case, as we need to do three successive operations. Firstly, we need to operate along submanifolds with boundaries and secondly to control certain constants in each step in order that the homeomorphisms corresponding to the operations stay close to the identity. For the first operation, we need some quatitative estimates in the annulus theorem, which are easily obtained by varying the standard proof of the annulus theorem in dimension 2 based on Schönflies' Theorem. The only thing that matters about quantitative estimates in that they exist. For the sake of completeness we include a full proof.

Fix three points $v_0, v_1, v_2 \in \mathbb{R}^2$ such that for $i \neq j$, $d(v_i, v_j) = 1$, and denote by $\triangle$ the 2-cell consisting of the points lying within the triangle $\triangle v_0 v_1 v_2$. Suppose also that the barycenter of $\triangle$ lies at the origin, so that for all $\lambda > 0$, $\lambda\triangle$ and $\triangle$ are concentric triangles, the former with side lengths λ.

Lemma 2.4: *Let $\phi : (1 - 2\eta)\triangle \to \triangle$ be a homeomorphic embedding satisfying*

$$\sup_{x \in (1-2\eta)\triangle} d(x, \phi(x)) < \eta/100,$$

where $\eta < 1/1000$. Then there is a homeomorphism $\psi : \triangle \to \triangle$ that is the identity outside of $(1 - \eta)\triangle$, with $\sup_{x \in \triangle} d(x, \psi(x)) < 100\eta$, and such that $\psi \circ \phi|_{(1-2\eta)\triangle} = \mathrm{id}$.

Proof. Let $\partial(1-\eta)\triangle$ be the boundary of $(1-\eta)\triangle$ and pick a finite set of points $\mathbb{F}$ containing $(1 - \eta)v_0, (1 - \eta)v_1, (1 - \eta)v_2$ and lying in $\partial(1 - \eta)\triangle$, such that when $\mathbb{F}$ is equipped with the circular order obtained from going counterclockwise around $\partial(1-\eta)\triangle$, we have $d(x,x^+)\in]20\eta,21\eta[$ for all $x\in\mathbb{F}$. As $\triangle$ is equilateral, $d(x,y)>20\eta$ for all $x\neq y$ in $\mathbb{F}$.

Let now $C=\phi[\partial(1-2\eta)\triangle]$ be the image of the boundary of $(1-2\eta)\triangle$, so C is a simple closed curve. Choose also for each $x\in\mathbb{F}$ a point $\hat{x}\in C$ such that the distance $d(x,\hat{x})$ is minimal. Since $\sup_{x\in(1-2\eta)\triangle} d(x,\phi(x))<\eta/100$ and

$$\eta/3<d(x,\partial(1-2\eta)\triangle)<(2\eta)/3$$

for all $x\in\partial(1-\eta)\triangle$, also $d(x,\hat{x})<\eta$ and $d(C,\partial(1-\eta)\triangle)>\eta/4$.

For all $x\in\mathbb{F}$, denote by α_x the straight (oriented) line segment from x to $\hat{x}$ and by β_x the straight line segment from x to x^+. We also let γ'_x be the shortest path in $\partial(1-2\eta)\triangle$ from $\phi^{-1}(\hat{x})$ to $\phi^{-1}(\widehat{x^+})$ and put $\gamma_x=\phi[\gamma'_x]$.

By definition of $\hat{x}$, α_x intersects C exactly in $\hat{x}$, intersects $\partial(1-\eta)\triangle$ in exactly x, and therefore α_x and γ_y intersect only if $y=x^-$ or $y=x$. Similarly, none of the paths β_x and γ_y intersect as they lie in $\partial(1-\eta)\triangle$ and C respectively. Therefore, for any $x\in\mathbb{F}$, $\mathcal{C}_x=\alpha_x\centerdot\gamma_x\centerdot\bar{\alpha}_{x^+}\centerdot\bar{\beta}_x$ is a simple closed curve beginning and ending at x. Here $\bar{\alpha}$ denotes the reverse path of α and $\centerdot$ the concatenation of paths. By the Schönflies Theorem, $\mathbb{R}^2\setminus\mathcal{C}_x$ has exactly two components, one unbounded and the other U_x bounded, homeomorphic with $\mathbb{R}^2$ and with boundary $\mathcal{C}_x$. Moreover, as the diameter of $\mathcal{C}_x$ is bounded by 30η, $\mathcal{C}_x$ intersects $\partial(1-\eta)\triangle$ in exactly β_x, and the diameter of $\partial(1-\eta)\triangle\setminus\beta_x$ is $1-\eta>30\eta$, this means that $\partial(1-\eta)\triangle\setminus\beta_x$ lies in the unbounded component. Therefore, if $R_x=\overline{U}_x=U_x\cup\mathcal{C}_x$, we have for $x\neq y$

$$R_x\cap R_y=\begin{cases}\emptyset & \text{if } y\neq x^+ \text{ and } y\neq x^-\\ \alpha_y & \text{if } y=x^+\\ \alpha_x & \text{if } y=x^-\end{cases}$$

We can now define $\psi:\triangle\to\triangle$ by letting $\psi=\phi^{-1}$ on $\phi[(1-2\eta)\triangle]$, $\psi=\mathrm{id}$ on $\triangle\setminus(1-\eta)\triangle$, and, moreover, along the boundaries of R_x construct ψ as follows: $\psi[\alpha_x]$ is the straight line segment from x to $\phi^{-1}(\hat{x})$, $\psi[\gamma_x]=\gamma'_x$, and $\psi[\beta_x]=\beta_x$. Then

$$\psi[\mathcal{C}_x]=\psi[\alpha_x\centerdot\gamma_x\centerdot\bar{\alpha}_{x^+}\centerdot\bar{\beta}_x]=\psi[\alpha_x]\centerdot\psi[\gamma_x]\centerdot\overline{\psi[\alpha_{x^+}]}\centerdot\overline{\psi[\beta_x]}=\psi[\alpha_x]\centerdot\gamma'_x\centerdot\overline{\psi[\alpha_{x^+}]}\centerdot\overline{\beta}_x$$

is the boundary of a region K_x homeomorphic to the unit disk D^2 and hence, by Alexander's Lemma, the homeomorphism ψ from $\mathcal{C}_x=\alpha_x\centerdot\gamma_x\centerdot\bar{\alpha}_{\hat{x}}\centerdot\bar{\beta}_x$ to $\psi[\alpha_x]\centerdot\gamma'_x\centerdot\overline{\psi[\alpha_{x^+}]}\centerdot\overline{\beta}_x$ extends to the regions that they bound, i.e., to a homeomorphism of R_x with K_x. This finishes the description of ψ and it therefore only remains to proof that $\sup_{x\in\triangle} d(x,\psi(x)) < 100\eta$. Since $\psi = \phi^{-1}$ on $\phi[(1-2\eta)\triangle]$ and $\psi = \mathrm{id}$ on $\triangle \setminus (1-\eta)\triangle$ it is enough to consider what ψ does to $x \in (1-\eta)\triangle \setminus \phi[(1-2\eta)\triangle] \subseteq \bigcup_{x\in\mathbb{F}} R_x$. Now, $\psi[R_x] = K_x$ for all $x \in \mathbb{F}$, and hence it is enough to show that no points in R_x and in K_x are more than 100η apart. But $\mathrm{diam}(R_x) < 30\eta$ and $\mathrm{diam}(K_x) < 40\eta$, while $R_x \cap K_x \neq \emptyset$, which gives the desired result. This finishes the proof. ∎

2.5. Patching along a triangulation of a compact 2-manifold. As $\mathrm{Homeo}(M)$ is a separable complete metric group, it is not covered by countably many nowhere dense sets (this is the Baire category theorem) and hence W must be dense in some non-empty open set, whereby $W^{-1}W = W^2$ is dense in some neighbourhood of the identity in $\mathrm{Homeo}(M)$. So fix some $\eta_1 > 0$ such that W^2 is dense in

$$V_{\eta_1} = \{g \in \mathrm{Homeo}(M) \mid d_\infty(g,\mathrm{id}) < \eta_1\}. \tag{1}$$

It is a well-known fact, first proved rigorously by Tibor Radó [Ra24], that any compact 2-manifold can be triangulated. So from now on, we assume that M is a fixed compact 2-manifold and we pick a triangulation $\{T_1,\ldots,T_m\}$ of M with corresponding homeomorphisms $\chi_i : \triangle \to T_i$. By further triangulating each T_i, we can suppose that the diameter of T_i is less than $\eta_1/10$ for all i. Moreover, by first modifying the χ_i along each edge of $\triangle$ and then extending to the interior of $\triangle$ by Alexander's Lemma, we can suppose that the following holds. If $T_i = \chi_i[\triangle]$ and $T_j = \chi_j[\triangle]$ have an edge in common, then χ_i and χ_j agree along this edge, i.e., if $\chi_i(v_a) = \chi_j(v_\alpha)$ and $\chi_i(v_b) = \chi_j(v_\beta)$, then for all $t \in [0,1]$, $\chi_i(tv_a + (1-t)v_b) = \chi_j(tv_\alpha + (1-t)v_\beta)$.

Lemma 2.5: *For all $0 < \eta < 1$, if $h \in \mathrm{Homeo}(M)$ has support contained in $\bigcup_{i=1}^m \chi_i[(1-\eta)\triangle]$, then $h \in W^{20}$.*

Proof. Let $y_i = \chi_i(\vec{0})$ and choose $\epsilon > 0$ such that $\overline{B}(y_i,\epsilon) \subseteq \chi_i[(1-\eta)\triangle]$ for all $i \leq m$. By Lemma 2.2, we can find some $0 < \delta < \epsilon$ and $z_i \in B(y_i,\epsilon)$ such that if $g \in \mathrm{Homeo}(M)$ has support contained in $\bigcup_{i=1}^m \overline{B}(z_i,\delta)$, then $g \in W^{16}$.

As W^2 is dense in V_{η_1}, we can find an $f \in W^2$ such that for every $i \leq m$, $f[\chi_i[(1-\eta)\triangle]] \subseteq \overline{B}(z_i,\delta)$ and thus if h is given as in the statement of the lemma, $\mathrm{supp}(fhf^{-1}) = f[\mathrm{supp}(h)] \subseteq \bigcup_{i=1}^m \overline{B}(z_i,\epsilon)$ and thus $g = fhf^{-1} \in W^{16}$, whence $h \in W^{20}$. ∎

Lemma 2.6: *Let $\delta,\eta>0$, $\eta<1/1000$ be such that for $i\leq m$ and $x,y\in\triangle$,*

$$d(x,y)<100\eta\rightarrow d(\chi_i(x),\chi_i(y))<\delta.$$

Then there is an $\alpha>0$ such that for all $g\in V_\alpha$ there is $\psi\in V_\delta\cap W^{20}$ whose support is contained in $\bigcup_{i=1}^m\chi_i[(1-\eta)\triangle]$ and such that for all $i\leq m$,

$$\psi\circ g|_{\chi_i[(1-2\eta)\triangle]}=\mathrm{id}.$$

Proof. Fix δ and η as in the lemma. Then for any continuous $\phi:\triangle\to\triangle$ such that $\sup_{x\in\triangle}d(x,\phi(x))<100\eta$, we have for every $i\leq m$,

$$\sup_{y\in T_i}d(y,\chi_i\circ\phi\circ\chi_i^{-1}(y))=\sup_{x\in\triangle}d(\chi_i(x),\chi_i\circ\phi(x))<\delta.$$

Now pick some $\alpha>0$ such that for $g\in V_\alpha$ and $i\leq m$, we have

$$g\circ\chi_i[(1-2\eta)\triangle]\subseteq\chi_i[\triangle]=T_i,$$

whereby $\chi_i^{-1}\circ g\circ\chi_i:(1-2\eta)\triangle\to\triangle$, and such that

$$\sup_{x\in(1-2\eta)\triangle}d(x,\chi_i^{-1}\circ g\circ\chi_i(x))<\eta/100.$$

By Lemma 2.4 we can find some homeomorphism $\psi_i:\triangle\to\triangle$ that is the identity outside of $(1-\eta)\triangle$, that satisfies the estimate $\sup_{x\in\triangle}d(x,\psi_i(x))<100\eta$, and

$$\psi_i\circ\chi_i^{-1}\circ g\circ\chi_i|_{(1-2\eta)\triangle}=\mathrm{id}.$$

This implies that for each $i\leq m$, $\chi_i\circ\psi_i\circ\chi_i^{-1}:T_i\to T_i$ is a homeomorphism that is the identity outside of $\chi_i[(1-\eta)\triangle]$, $\sup_{x\in T_i}d(x,\chi_i\circ\psi_i\circ\chi_i^{-1}(x))<\delta$, and

$$\chi_i\circ\psi_i\circ\chi_i^{-1}\circ g|_{\chi_i[(1-2\eta)\triangle]}=\mathrm{id}.$$

We can thus define $\psi=\bigcup_{i=1}^m\chi_i\circ\psi_i\circ\chi_i^{-1}\in\mathrm{Homeo}(M)$ and notice that $\psi\in V_\delta$ and $\psi\circ g|_{\chi_i[(1-2\eta)\triangle]}=\mathrm{id}$ for every $i\leq m$. We see that ψ has its support contained within the set $\bigcup_{i=1}^m\chi_i[(1-\eta)\triangle]$ and thus, by Lemma 2.5, ψ belongs to W^{20}. ∎

Fix some $0<\tau<1/100$. We now define the following set of points in $\triangle$ (see figure 1): For distinct $i,j=0,1,2$, we put $w_{ij}=(1-10\tau)v_i+10\tau v_j$, $w_{ij}^+=(1-9\tau)v_i+9\tau v_j$, $u_{ij}=(1-\tau)w_{ij}$ and $u_{ij}^+=(1-\tau)w_{ij}^+$. So $w_{ij},w_{ij}^+\in\partial\triangle$, while $u_{ij},u_{ij}^+\in\partial(1-\tau)\triangle$.

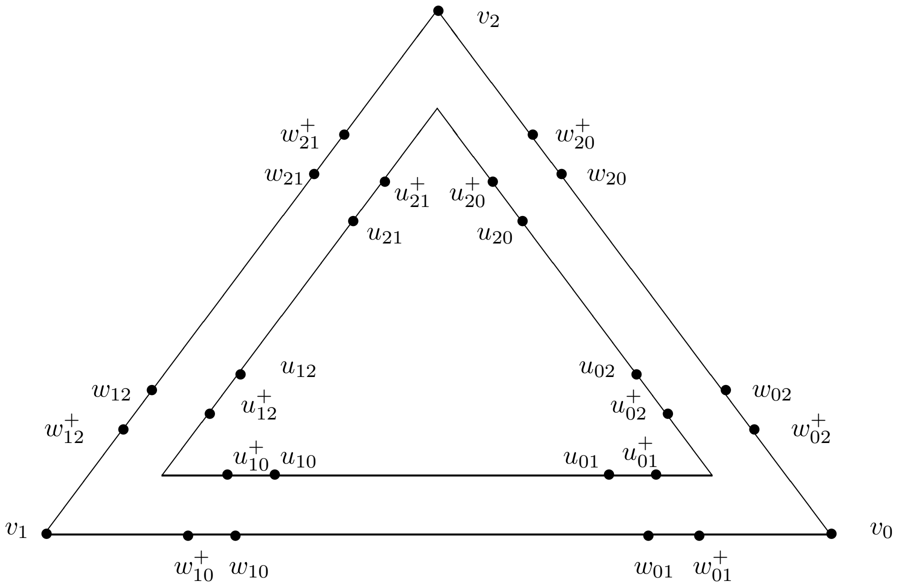

Figure 1

We also define a number of paths as follows (see figure 2):

- α_{ij} is the straight line segment from u_{ij} to w_{ij}.
- β_{ij} is the straight line segment from w_{ij} to w_{ij}^+.
- γ_{ij} is the straight line segment from u_{ij}^+ to w_{ij}^+.
- ζ_{ij} is the straight line segment from u_{ij} to u_{ij}^+.
- κ_{ij} is the straight path from w_{ij} to w_{ji}.
- ω_{ij} is the straight path from u_{ij} to u_{ji}.
- ξ_0 is the shortest path in $\partial(1-\tau)\triangle$ from u_{02}^+ to u_{01}^+.
- ξ_1 is the shortest path in $\partial(1-\tau)\triangle$ from u_{10}^+ to u_{12}^+.
- ξ_2 is the shortest path in $\partial(1-\tau)\triangle$ from u_{21}^+ to u_{20}^+.
- θ_0 is the shortest path in $\partial\triangle$ from w_{02}^+ to w_{01}^+.
- θ_1 is the shortest path in $\partial\triangle$ from w_{10}^+ to w_{12}^+.
- θ_2 is the shortest path in $\partial\triangle$ from w_{21}^+ to w_{20}^+.

We thus see that

$$\mathcal{C}_{ij} = \kappa_{ij} \centerdot \overline{\alpha}_{ji} \centerdot \omega_{ji} \centerdot \alpha_{ij}$$

is a simple closed curve bounding a closed region $R_{ij} = R_{ji} \subseteq \triangle$,

$$\mathcal{C}^+_{ij} = \overline{\beta}_{ij} \centerdot \kappa_{ij} \centerdot \beta_{ji} \centerdot \overline{\gamma}_{ji} \centerdot \overline{\zeta}_{ji} \centerdot \omega_{ji} \centerdot \zeta_{ij} \centerdot \gamma_{ij}$$

is a simple closed curve bounding a closed region $R^+_{ij} = R^+_{ji} \subseteq \triangle$ that contains R_{ij}.

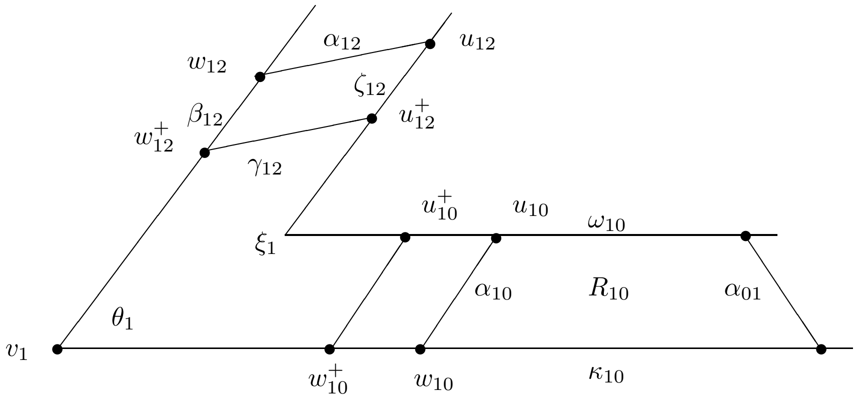

Figure 2

Notice however that the preceding definitions depend on the choice of τ, which is also the case for the following lemma.

Lemma 2.7: *If* $\phi \in \mathrm{Homeo}(M)$ *has support contained in* $\bigcup_{l=1}^m \bigcup_{0\leq i<j\leq 2} \chi_l[R^+_{ij}]$, *then* $\phi \in W^{20}$.

Proof. We notice that for distinct l, l', $\chi_l[R^+_{ab}] \cap \chi_{l'}[R^+_{a'b'}] \neq \emptyset$ if and only if the triangles T_l and $T_{l'}$ have the edge $\chi_l[\overline{v_a v_b}] = \chi_{l'}[\overline{v_{a'} v_{b'}}]$ in common. Moreover, in this case, the set $\chi_l[R^+_{ab}] \cup \chi_{l'}[R^+_{a'b'}]$ is homeomorphic to the unit disk D^2 and is contained in an open set homeomorphic to $\mathbb{R}^2$.

So let $A_1, \ldots, A_{\frac{3m}{2}}$ be an enumeration of all the closed sets $\chi_l[R^+_{ab}] \cup \chi_{l'}[R^+_{a'b'}]$ with $\chi_l[R^+_{ab}]$ and $\chi_{l'}[R^+_{a'b'}]$ overlapping and let $U_i \subseteq M$ be an open set containing A_i, homeomorphic to $\mathbb{R}^2$. We can suppose that the U_i are all pairwise disjoint.

Moreover, as the diameter of each T_j is at most $\eta_1/10$, the diameter of each A_i is at most $\eta_1/5$.

The proof is now very much the same as the proof of Lemma 2.5. Let $y_i \in A_i$ and choose $0 < \epsilon < \eta_1/5$ such that $\overline{B}(y_i, \epsilon) \subseteq U_i$ for all $i \leq m$. By Lemma 2.2, we can find some $0 < \delta < \epsilon$ and $z_i \in B(y_i, \epsilon)$ such that if $g \in \mathrm{Homeo}(M)$ has support contained in $\bigcup_{i=1}^m \overline{B}(z_i, \delta)$ then $g \in W^{16}$.

As W^2 is dense in V_{η_1}, we can find an $f \in W^2$ such that for every $i \leq 3m/2$, $f[A_i] \subseteq \overline{B}(z_i, \delta)$ and thus if ϕ is given as in the statement of the lemma,

$$\mathrm{supp}(f\phi f^{-1}) = f[\mathrm{supp}(\phi)] \subseteq \bigcup_{i=1}^{m} \overline{B}(z_i, \epsilon),$$

and thus $g = f\phi f^{-1} \in W^{16}$, whence $\phi \in W^{20}$. ∎

Lemma 2.8: *There is a $\nu > 0$ such that if $g \in V_\nu$ and g is the identity on*

$$\bigcup_{i=1}^{m} \chi_i[(1-\tau)\triangle],$$

then there is a $\phi \in W^{20}$ such that $\phi \circ g$ is the identity on

$$\bigcup_{i=1}^{m} \chi_i[(1-\tau)\triangle] \cup \bigcup_{l=1}^{m} \bigcup_{0\leq i<j\leq 2} \chi_l[R_{ij}].$$

Proof. Consider the closed set $M_0 = M \setminus \mathrm{Int}(\bigcup_{i=1}^m \chi_i[(1-\tau)\triangle])$ and the closed subgroup $H = \{g \in \mathrm{Homeo}(M) : g|_{\bigcup_{i=1}^m \chi_i[(1-\tau)\triangle]} = \mathrm{id}\}$. Assume that T_l and $T_{l'}$ have an edge in common, i.e., $\chi_l(v_a) = \chi_{l'}(v_{a'})$ and $\chi_l(v_b) = \chi_{l'}(v_{b'})$ for some a, a', b, b'. Then $\chi_l[R_{ab}] \cup \chi_{l'}[R_{a'b'}] \subseteq \mathrm{Int}_{M_0}(\chi_l[R^+_{ab}] \cup \chi_{l'}[R^+_{a'b'}])$. Therefore, we can find some $\nu > 0$, not depending on the particular choice of l, l', a, a', b, b', such that for all such choices of l, l', a, a', b, b' and $g \in V_\nu \cap H$ we have

$$g[\chi_l[R_{ab}] \cup \chi_{l'}[R_{a'b'}]] \subseteq \mathrm{Int}_{M_0}(\chi_l[R^+_{ab}] \cup \chi_{l'}[R^+_{a'b'}]). \tag{2}$$

Fix some $g \in V_\nu \cap H$.

Assume now that $\chi_l[\triangle]$ and $\chi_k[\triangle]$ have an edge in common. For concreteness we can suppose that, e.g., $\chi_l(v_0) = \chi_k(v_1)$ and $\chi_l(v_1) = \chi_k(v_2)$. As the covering mappings χ_i were supposed to agree along their edges, this implies that $\chi_l[\beta_{01}] = \chi_k[\beta_{12}]$, $\chi_l[\kappa_{01}] = \chi_k[\kappa_{12}]$, and $\chi_l[\beta_{10}] = \chi_k[\beta_{21}]$. Also, as $g \in H$, g is the identity on the paths $\chi_l[\zeta_{01}], \chi_l[\omega_{01}], \chi_l[\zeta_{10}], \chi_k[\zeta_{12}], \chi_k[\omega_{12}]$ and $\chi_k[\zeta_{21}]$.

By consequence, $\chi_l[\zeta_{01}] \centerdot \chi_l[\gamma_{01}] \centerdot \chi_k[\overline{\gamma}_{12}] \centerdot \chi_k[\overline{\zeta}_{12}]$ and $\chi_l[\alpha_{01}] \centerdot \chi_k[\overline{\alpha}_{12}]$ are paths from $\chi_l(u_{01})$ to $\chi_k(u_{12})$ only intersecting in their endpoints. Similarly,

$\chi_l[\zeta_{10}] \centerdot \chi_l[\gamma_{10}] \centerdot \chi_k[\overline{\gamma}_{21}] \centerdot \chi_k[\overline{\zeta}_{21}]$ and $\chi_l[\alpha_{10}] \centerdot \chi_k[\overline{\alpha}_{21}]$ are paths from $\chi_l(u_{10})$ to $\chi_k(u_{21})$ only intersecting in their endpoints. This shows that

$$\mathcal{K} = \chi_l[\zeta_{01}] \centerdot \chi_l[\gamma_{01}] \centerdot \chi_k[\overline{\gamma}_{12}] \centerdot \chi_k[\overline{\zeta}_{12}] \centerdot \chi_k[\alpha_{12}] \centerdot \chi_l[\overline{\alpha}_{01}]$$

is a simple closed curve and thus, by the Schönflies Theorem, bounds a region A homeomorphic to the unit disk D^2. Similarly,

$$\mathcal{K}' = \chi_l[\zeta_{10}] \centerdot \chi_l[\gamma_{10}] \centerdot \chi_k[\overline{\gamma}_{21}] \centerdot \chi_k[\overline{\zeta}_{21}] \centerdot \chi_k[\alpha_{21}] \centerdot \chi_l[\overline{\alpha}_{10}]$$

is a simple closed curve and thus bounds a region A' homeomorphic to the unit disk D^2.

Now, as $\chi_l[\alpha_{01}] \centerdot \chi_k[\overline{\alpha}_{12}] \subseteq \chi_l[R_{01}] \cup \chi_k[R_{12}]$, by condition 2 on g,

$$g[\chi_l[\alpha_{01}] \centerdot \chi_k[\overline{\alpha}_{12}]] \subseteq \mathrm{Int}_{\mathrm{M}_0}(\chi_l[R^+_{01}] \cup \chi_k[R^+_{12}])$$

and hence intersects $\chi_l[\zeta_{01}] \centerdot \chi_l[\gamma_{01}] \centerdot \chi_k[\overline{\gamma}_{12}] \centerdot \chi_k[\overline{\zeta}_{12}]$ only in their common endpoints. Thus,

$$\mathcal{L} = \chi_l[\zeta_{01}] \centerdot \chi_l[\gamma_{01}] \centerdot \chi_k[\overline{\gamma}_{12}] \centerdot \chi_k[\overline{\zeta}_{12}] \centerdot g[\chi_k[\alpha_{12}]] \centerdot g[\chi_l[\overline{\alpha}_{01}]]$$

is a simple closed curve bounding a region B homeomorphic to D^2. Similarly,

$$\mathcal{L}' = \chi_l[\zeta_{10}] \centerdot \chi_l[\gamma_{10}] \centerdot \chi_k[\overline{\gamma}_{21}] \centerdot \chi_k[\overline{\zeta}_{21}] \centerdot g[\chi_k[\alpha_{21}]] \centerdot g[\chi_l[\overline{\alpha}_{10}]]$$

bounds a region B' homeomorphic to D^2.

We now have two decompositions of $\chi_l[R^+_{01}] \cup \chi_k[R^+_{12}]$.

(1) $A \cup \big[\chi_l[R_{01}] \cup \chi_k[R_{12}]\big] \cup A'$.
(2) $B \cup g[\chi_l[R_{01}] \cup \chi_k[R_{12}]] \cup B'$.

Here A and $\chi_l[R_{01}] \cup \chi_k[R_{12}]$ overlap along the edge

$$\chi_l[\alpha_{01}] \centerdot \chi_k[\overline{\alpha}_{12}],$$

$\chi_l[R_{01}] \cup \chi_k[R_{12}]$ and A' overlap along $\chi_l[\alpha_{10}] \centerdot \chi_k[\overline{\alpha}_{21}]$, while $A \cap A' = \emptyset$. Similarly, B and $g[\chi_l[R_{01}] \cup \chi_k[R_{12}]]$ overlap along the edge

$$g[\chi_l[\alpha_{01}]] \centerdot g[\chi_k[\overline{\alpha}_{12}]],$$

$g[\chi_l[R_{01}] \cup \chi_k[R_{12}]]$ and B' overlap along $g[\chi_l[\alpha_{10}]] \centerdot g[\chi_k[\overline{\alpha}_{21}]]$, while $B \cap B' = \emptyset$.

We can now define a homeomorphism

$$\varphi_{lk} : \chi_l[R^+_{01}] \cup \chi_k[R^+_{12}] \to \chi_l[R^+_{01}] \cup \chi_k[R^+_{12}],$$

by first setting $\varphi_{lk} = g^{-1}$ on $g[\chi_l[R_{01}] \cup \chi_k[R_{12}]]$, and then let φ_{lk} send B to A, while fixing each point of $\chi_l[\zeta_{01}] \centerdot \chi_l[\gamma_{01}] \centerdot \chi_k[\overline{\gamma}_{12}] \centerdot \chi_k[\overline{\zeta}_{12}]$ and be g^{-1} on $g[\chi_l[\alpha_{01}] \centerdot \chi_k[\overline{\alpha}_{12}]]$. Similarly for B' and A'.

This can be done for all pairs of χ_l and χ_k with a common edge, and we thus produce homeomorphisms φ_{lk} on all of the regions, similar to $\chi_l[R^+_{01}] \cup \chi_k[R^+_{12}]$, that fix each point of the boundary curve

$$\chi_l[\omega_{10}] \centerdot \chi_l[\zeta_{10}] \centerdot \chi_l[\gamma_{01}] \centerdot \chi_k[\overline{\gamma}_{12}] \centerdot \chi_k[\overline{\zeta}_{12}] \centerdot \chi_k[\omega_{12}] \centerdot \chi_k[\zeta_{21}] \centerdot \chi_k[\gamma_{21}] \centerdot \chi_l[\overline{\gamma}_{10}] \centerdot \chi_l[\overline{\zeta}_{10}].$$

Pasting all of these φ_{lk} together and extending to all of M by setting $\phi = \mathrm{id}$ elsewhere, we obtain a homeomorphism $\phi \in \mathrm{Homeo}(M)$ whose support is contained in $\bigcup_{l=1}^m \bigcup_{0\le i<j\le 2} \chi_l[R^+_{ij}]$, while being the inverse of g on $\bigcup_{l=1}^m \bigcup_{0\le i<j\le 2} \chi_l[R_{ij}]$. By Lemma 2.7, $\phi \in W^{20}$, which finishes the proof. ∎

We are now ready to finish the proof of Theorem 1.1 using the preceding sequence of lemmas.

Proof. Let $y_1, \dots, y_p \in M$ be the vertices of the triangulation and choose for each $i \le p$ a neighbourhood U_i of y_i homeomorphic to $\mathbb{R}^2$. Find also $0 < \epsilon < \eta_1$ such that $\overline{B}(y_i, \epsilon) \subseteq U_i$ for all i. By Lemma 2.2, there are $0 < \delta_0 < \epsilon$, $z_i \in B(y_i, \epsilon)$, such that if $g \in \mathrm{Homeo}(M)$ has support contained in $\bigcup_{i=1}^p \overline{B}(z_i, \delta_0)$, then $g \in W^{16}$. As $y_i, z_i \in U_i \simeq \mathbb{R}^2$, we can, as W^2 is dense in V_{η_1}, find some $h_0 \in W^2$ such that $h_0(y_i) \in U'_i \subseteq \overline{B}(z_i, \delta_0)$, where U'_i is a neighbourhood of z_i homeomorphic to $\mathbb{R}^2$. Therefore, there is some $g_0 \in W^{16}$ such that $g_0 h_0(y_i) = z_i$. This shows that if $f \in \mathrm{Homeo}(M)$ has support contained in $U = (g_0h_0)^{-1}[\bigcup_{i=1}^p]$, then $(g_0h_0)^{-1} f (g_0h_0)$ has support contained in $\bigcup_{i=1}^p B(z_i, \delta_0)$ and hence belongs to W^{16}. So f belongs to W^{52}. We notice that U is an open set containing $y_1, \dots, y_p$.

Recall now the definition of the paths α_{ij}, β_{ij}, etc. and also the fact that these paths all depend on the choice of $0 < \tau < 1$. For a fixed choice of τ, we define the following simple closed curves in $\triangle$

$$\begin{aligned} \mathcal{F}^\tau_0 &= \beta_{02} \centerdot \theta_0 \centerdot \overline{\beta}_{01} \centerdot \overline{\alpha}_{01} \centerdot \zeta_{01} \centerdot \overline{\xi}_0 \centerdot \overline{\zeta}_{02} \centerdot \alpha_{02}, \\ \mathcal{F}^\tau_1 &= \beta_{10} \centerdot \theta_1 \centerdot \overline{\beta}_{12} \centerdot \overline{\alpha}_{12} \centerdot \zeta_{12} \centerdot \overline{\xi}_1 \centerdot \overline{\zeta}_{10} \centerdot \alpha_{10}, \\ \mathcal{F}^\tau_2 &= \beta_{21} \centerdot \theta_2 \centerdot \overline{\beta}_{20} \centerdot \overline{\alpha}_{20} \centerdot \zeta_{20} \centerdot \overline{\xi}_2 \centerdot \overline{\zeta}_{21} \centerdot \alpha_{21}. \end{aligned} \tag{3}$$

Moreover, we let $F^\tau_0, F^\tau_1, F^\tau_2$ be the closed regions that they enclose. We notice that F^τ_i converges in the Vietoris topology to $\{v_i\}$ when $\tau \to 0$, and thus for some $\tau > 0$, we have for all $i = 0, 1, 2$ and $l = 1, \dots, m$, $\chi_l[F^\tau_i] \subseteq U$. So fix this τ and denote F^τ_i by F_i. We notice that

$$\triangle = (1-\tau)\triangle \cup \bigcup_{0\le i<j\le 2} R_{ij} \cup \bigcup_{i=0,1,2} F_i.$$

By consequence, if $f \in \mathrm{Homeo}(M)$ is the identity on

$$\bigcup_{i=1}^{m} \chi_i[(1-\tau)\triangle] \cup \bigcup_{l=1}^{m} \bigcup_{0\leq i<j\leq 2} \chi_l[R_{ij}],$$

then f has support contained in $\bigcup_{l=1}^{m} \bigcup_{i=0,1,2} \chi_l[F_i] \subseteq U$, and hence $f \in W^{52}$.

Find now a $\nu > 0$ as in the statement of Lemma 2.8. Then if $g \in V_\nu$ and g is the identity on $\bigcup_{i=1}^{m} \chi_i[(1-\tau)\triangle]$, then there is a $\phi \in W^{20}$ such that $\phi \circ g$ is the identity on

$$\bigcup_{i=1}^{m} \chi_i[(1-\tau)\triangle] \cup \bigcup_{l=1}^{m} \bigcup_{0\leq i<j\leq 2} \chi_l[R_{ij}],$$

and hence belongs to W^{52}. But then also $g \in W^{72}$.

Fix $\delta < \nu/2$ and find an $\eta > 0$ satisfying $\eta < 1/1000$, $\eta < \nu/2$, and such that for $i \leq m$ and $x, y \in \triangle$,

$$d(x,y) < 100\eta \to d(\chi_i(x), \chi_i(y)) < \delta.$$

By Lemma 2.6, we can find an $0 < \alpha < \nu/2$ such that for all $h \in V_\alpha$ there is $\psi \in V_\delta \cap W^{20}$ such that for all $i \leq m$,

$$\psi \circ h|_{\chi_i[(1-2\eta)\triangle]} = \mathrm{id}.$$

In particular, $\psi \circ h \in V_\delta V_\alpha \subseteq V_{\delta+\alpha} \subseteq V_\nu$ and is the identity on $\bigcup_{i=1}^{m} \chi_i[(1-\tau)\triangle]$, whereby $\psi \circ h \in W^{72}$ and $h \in W^{92}$. This shows that $V_\alpha \subseteq W^{92}$ and thus W^{92} contains an open neighbourhood of the identity in $\mathrm{Homeo}(M)$, and hence we have proved that $\mathrm{Homeo}(M)$ is Steinhaus, which finishes the proof of Theorem 1.1. ∎

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